SOLUTION - A Simple Puzzle 3

$$\mathbf{A} = \begin{pmatrix} 6 & 0 & 8 \\ -1 & 7 & 2 \\ 0 & 0 & 10 \end{pmatrix}$$

$$\det(\mathbf{A}) = 420$$

YES, that's right - 420! At the University of Pennsylvania, the MATH 420 class is none other than *Ordinary Differential Equations*. Hence April 20 is known as Ordinary Differential Equations (ODE) Day at Penn. It's a time when students of all majors get together with truckloads of pizza and Red Bull and solve ODEs till the wee hours of the morning... and I gotta tell ya, it's madness, MADNESS I say!!!

Yeah,... OK, I just totally made all that shit up. There's no such thing as ODE Day at Penn - but there *should* be. Of course 420 has a certain cultural relevance for some people, but as far as this puzzle is concerned, that's just pure... uh... coincidence.

BACKGROUND AND MOTIVATION: Let me present a brief overview of the matrix exponential as it relates to some basic concepts of ODEs.

Consider the following system of first order homogeneous linear equations with constant coefficients:

$$\begin{aligned} x_1' &= a_{11}x_1 + \cdots + a_{1n}x_n \\ &\cdot \\ &\cdot \\ &\cdot \\ x_n' &= a_{n1}x_1 + \cdots + a_{nn}x_n \end{aligned} \tag{1}$$

where $a_{11}, ..., a_{nn}$ are constants, $x_1, ..., x_n$ represent functions of an independent variable t, and differentiation with respect to t is denoted by a prime. It is often convenient to write such systems of equations in matrix notation. If we let $\mathbf{x}$ and $\mathbf{x}'$ denote the vectors

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}, \qquad \mathbf{x}' = \begin{bmatrix} x_1' \\ \cdot \\ \cdot \\ \cdot \\ x_n' \end{bmatrix} \tag{2}$$

and let $\mathbf{A}$ denote the matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & \cdot & \cdot & \cdot & a_{1n} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ a_{n1} & \cdot & \cdot & \cdot & a_{nn} \end{pmatrix} \tag{3}$$

then the system of equations (1) can be written more concisely as

$$\mathbf{x}' = \mathbf{A}\mathbf{x}. \tag{4}$$

A solution of (4) is a vector-valued function that satisfies the equation for all values of t in some interval (remember all that crap about eigenvalues and eigenvectors?). Now there always exists what is called a *fundamental set of solutions* to (4). Since we are considering a system of n equations, in this case, a fundamental set of solutions would consist of n linearly independent functions. Each solution of (4) can be be written as a unique linear combination of the n functions. In other words, the fundamental set of solutions comprises a *basis* for the n-dimensional vector space of all solutions. For example, if $\mathbf{x}^{(1)}(t), ..., \mathbf{x}^{(n)}(t)$ forms a fundamental set of solutions for (4) on some interval, then

$$\mathbf{x} = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t) \tag{5}$$

is also a solution for any constants $c_1, ..., c_n$. If we consider the constants to be arbitrary, then (5) is called a *general solution*. If we form the matrix

$$\mathbf{\Psi}(t) = \begin{pmatrix} x_1^{(1)}(t) & \cdot & \cdot & \cdot & x_1^{(n)}(t) \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ x_n^{(1)}(t) & \cdot & \cdot & \cdot & x_n^{(n)}(t) \end{pmatrix} \tag{6}$$

where the columns are the vectors $\mathbf{x}^{(1)}(t), ..., \mathbf{x}^{(n)}(t)$, then $\mathbf{\Psi}(t)$ is called a *fundamental matrix* for (4). If we let $\mathbf{c}$ represent the arbitrary constant vector with elements $c_1, ..., c_n$, then the general solution can be written as

$$\mathbf{x} = \mathbf{\Psi}(t)\mathbf{c}. \tag{7}$$

If we are considering an initial value problem (IVP)

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{x}^0 \tag{8}$$

where $\mathbf{x}^0$ is a given initial vector, then $\mathbf{c}$ must satisfy

$$\mathbf{\Psi}(t_0)\mathbf{c} = \mathbf{x}^0. \tag{9}$$

Since $\mathbf{\Psi}$ is formed from linearly independent vectors, it is *nonsingular* (*i.e.* it has an inverse) and we have

$$\mathbf{c} = \mathbf{\Psi}^{-1}(t_0)\mathbf{x}^0. \tag{10}$$

Substituting (10) into (7) we get

$$\mathbf{x} = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0)\mathbf{x}^0 \tag{11}$$

which is a solution of the IVP (8).

Note also that since every column of $\mathbf{\Psi}$ is a solution of (4), it can be easily confirmed that $\mathbf{\Psi}$ satisfies the matrix differential equation

$$\mathbf{\Psi}' = \mathbf{A}\mathbf{\Psi}. \tag{12}$$

There is a special kind of fundamental matrix which is often convenient to use which we will denote by $\mathbf{\Phi}(t)$. The columns of $\mathbf{\Phi}(t)$ are solutions that also satisfy the initial conditions

$$\mathbf{x}^{(i)}(t_0) = e^{(i)} \tag{13}$$

where $e^{(i)}$ represents the unit vector that has 1 in the *ith* position and zeros everywhere else. It follows that

$$\mathbf{\Phi}(t_0) = \mathbf{I} = \mathbf{\Phi}^{-1}(t_0) \tag{14}$$

where $\mathbf{I}$ represents the identity matrix. Hence (11) can now be written as

$$\mathbf{x} = \mathbf{\Phi}(t)\mathbf{x}^0. \tag{15}$$

The fundamental matrix $\mathbf{\Phi}$ is especially useful when trying to solve a system of differential equations subject to many different initial conditions since the solution for each set of initial conditions can be found through simple matrix multiplication.

OK, now let's backtrack a little to the single differential equation case. Recall that the solution to the IVP

$$x' = ax, \quad x(0) = x_0 \tag{16}$$

where a is a constant is

$$x = x_0 e^{at}. \tag{17}$$

Comparing the form of (17) with (15) might lead one to believe that $\mathbf{\Phi}(t)$ could possibly be written in exponential form $e^{\mathbf{A}t}$. The question then arises of how to go about defining $e^{\mathbf{A}t}$. Remember that the scalar exponential function can be represented as a power series

$$e^{at} = 1 + \sum_{n=1}^{\infty} \frac{a^n t^n}{n!}. \tag{18}$$

You might be thinking that the matrix exponential $e^{\mathbf{A}t}$ can be defined in a similar way, and you would be right. That's why they pay you the big bucks. The corresponding series is

$$e^{\mathbf{A}t} = \mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{A}^n t^n}{n!} \tag{19}$$

where every term is actually an $n \times n$ matrix. It can be shown that (18) converges for all t and can differentiated term by term to get

$$\frac{d}{dt} e^{\mathbf{A}t} = \mathbf{A} e^{\mathbf{A}t}. \tag{20}$$

Also note that when we set $t = 0$, we have

$$e^{\mathbf{A}t}|_{\,t=0} = \mathbf{I}. \tag{21}$$

So, for $t_0 = 0$, $\mathbf{\Phi}$ satisfies the same IVP as $e^{\mathbf{A}t}$

$$\mathbf{\Phi}' = \mathbf{A}\mathbf{\Phi}, \quad \mathbf{\Phi}(0) = \mathbf{I}. \tag{22}$$

Now there is a theorem which states that any such solution must be unique. Hence $e^{\mathbf{A}t}$ and $\mathbf{\Phi}(t)$ must be the same !

OK, that's enough review. Let's move on. The puzzle can be solved by using (20) and (21). First take the derivative

$$\frac{d}{dt} e^{\mathbf{A}t} = \begin{pmatrix} 6e^{6t} & 0 & 20e^{10t} - 12e^{6t} \\ 6e^{6t} - 7e^{7t} & 7e^{7t} & 14e^{7t} - 12e^{6t} \\ 0 & 0 & 10e^{10t} \end{pmatrix}. \tag{23}$$

If we set $t = 0$ then we have

$$\frac{d}{dt}e^{\mathbf{A}t}|_{\,t=0} = \begin{pmatrix} 6 & 0 & 8 \\ -1 & 7 & 2 \\ 0 & 0 & 10 \end{pmatrix} \tag{24}$$

and

$$e^{\mathbf{A}t}|_{\,t=0} = \mathbf{I}. \tag{25}$$

Hence

$$\mathbf{A} = \begin{pmatrix} 6 & 0 & 8 \\ -1 & 7 & 2 \\ 0 & 0 & 10 \end{pmatrix} \tag{26}$$

and

$$\det(\mathbf{A}) = 420. \tag{27}$$

The problem with the earlier version of the puzzle (since some of you asked) was that I simply took a matrix from one of my old (wrong) homework problems and I (without thinking) posted something that was not a "proper" matrix exponential (I got an 'A' in MATH 420,... honest).

For the final form of the puzzle, I got help from a mysterious benfactor who I will call "Boris". Boris came up with a couple of very specific suggestions for how to build $e^{\mathbf{A}t}$ and he explicitly presented the following:

$$e^{\mathbf{A}t} = \begin{pmatrix} e^{6t} & 0 & 2e^{10t} - 2e^{6t} \\ e^{6t} - e^{7t} & e^{7t} & 2e^{7t} - 2e^{6t} \\ 0 & 0 & e^{10t} \end{pmatrix}. \tag{28}$$

So many thanks to Boris and to all who participated in my latest installment of "C'mon, can't you make the puzzles a little harder?".

Till next time, here's your homework assignment: Show that $e^{\mathbf{T}^{-1}\mathbf{A}\mathbf{T}} = \mathbf{T}^{-1}e^{\mathbf{A}}\mathbf{T}$.